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Stokes phenomena and the monodromy deformation problem for the non-linear Schrödinger equation

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Abstract. Following Flaschka and Newell we have formulated the inverse problem for Painlevé IV, with the help of similarity variables. The Painlevé IV arises as the eliminant of the two second-order ordinary differential equations originating from the non-linear Schrödinger equation. We have obtained the asymptotic expansions near the singularities at 0 and ∞ of the complex eigenvalue plane. The corresponding analysis then displays Stokes phenomena. The monodromy matrices connecting the solution Y_j in the sector S_j to that in S_{j+1} are fixed in structure by the imposition of certain conditions. We then show that a deformation keeping the monodromy data fixed leads to the non-linear Schrödinger equation. At this point we may mention that, while Flaschka and Newell did not make any absolute determination of the Stokes parameter, our appraoch yields the values of the Stokes parameter in an explicit way, which in turn can determine the matrix connecting the solutions near 0 and ∞. Such a realisation was not possible in the approach of Flaschka and Newell. Lastly we show that the integral equation originating from the analyticity and asymptotic nature leads to the similarity solution previously determined by Boiti and Pempinelli.

1. Introduction

Recently two important but parallel theories have been developed for the complete analysis of the non-linear partial differential equation. One is the method of inverse scattering transforms [1] and the other is that of monodromy deformation [2]. Though several authors have enriched the subject of IST, the contributions to the field of monodromy deformation (MD) are relatively few. The only exhaustive approach is that of the Kyoto school, mainly led by Ueno and Date [3] and Jimbo and Miwa [4]. Another approach is that of Flaschka and Newell [5]. While the method of the Japanese school is relativity abstract, being based on infinite-dimensional Lie algebras, that of Flaschka and Newell (FN) is more concrete and oriented to special non-linear equations, through its connection to the special class of the Painlevé equation. One of the best points of the FN approach is that it exhibits very clearly how the asymptotic expansion can be used in conjunction with analyticity arguments to analyse Stokes phenomena, and hence the monodromy deformation problem. But here we can indicate a point of departure from the treatment of FN. In the paper of FN, the absolute determination of Stokes parameters was not possible, but here we show that, by recourse to a classical analysis of Sibuya [6], it is possible to find the explicit values of the Stokes constants. These values can then be utilised in the equations determining the matrix connecting the solution vector near 0 and ∞ for their determination (see equation (31) in § 4). In this connection it can be noted that almost all the integrable non-linear equations reduce to some Painlevé transcendents through the similarity variables. On the other hand, the non-linear Schrödinger equation reduces to a pair of coupled ordinary equations equivalent to the Painlevé IV as shown by Boiti and Pempinelli [7]. In this paper we want to apply the methodology of FN, slightly amended by incorporating the theory of Sibuya, to the case of Painlevé IV. At this point we may mention that, though the works of [3, 4] encompass all the Painlevé equations, the formalistic nature of their approach is quite difficult to appreciate in terms of the results of any particular equation. On the other hand, our approach is of a pedagogical nature and clearly indicates the ways and means of circumventing the difficulties encountered in a analysis of the monodromy deformation problem.

2. Formulation

The non-linear Schrödinger equation under consideration is

$$\mathrm{i}q_t - q_{xx} = \pm 2q^2q^*. \tag{1}$$

The AKNS inverse problem pertaining to equation (1) is

$$\begin{aligned} v_{1x} &= -\mathrm{i}\xi' v_1 + qv_2 \\ v_{2x} &= \mathrm{i}\xi' v_2 + rv_1 \end{aligned} \tag{2}$$

along with

$$\begin{aligned} v_{1t} &= Av_1 + Bv_2 \\ v_{2t} &= Cv_1 + Dv_2 \end{aligned} \tag{3}$$

where A, B, C, D are well known functions of q, r and ξ' and for NLSE we assume $q = r^*$. The similarity variable which can be found either by a Lie point symmetry analysis or by a scaling argument is given as

$$z = xt^{-1/2} \qquad q(x, t) = t^{1/2}\phi(x/t^{1/2}).$$

We then convert equation (1) to the ordinary non-linear coupled system of differential equations

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left(\phi_2 + \frac{\mathrm{i}z}{2}\phi\right) = \pm 2\phi^2\phi^*. \tag{4}$$

The main trick of FN is to convert the Lax pair, (2) and (3), to such variables, for which we set

$$v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \qquad \begin{aligned} v &= v(xt^{-1/2}, \xi' t^{1/2}) \\ &= v(z, \xi) \end{aligned}$$

so that we have

$$\begin{aligned} v_z^1 &= -\mathrm{i}\xi n^1 + \phi v^2 \\ v_z^2 &= \mathrm{i}\xi v^2 + \phi^* v^1 \end{aligned} \tag{5}$$

$$v^1_\xi=\left(4\mathrm{i}\xi+\frac{2\mathrm{i}\phi\phi^*}{\xi}-\mathrm{i}z\right)v^1+\left(-4\phi-\frac{2\mathrm{i}\phi_z}{\xi}+\frac{\phi_z}{\xi}\right)v^2$$
$$v^2_\xi=\left(-4\phi^*+\frac{2\mathrm{i}\phi^*_z}{\xi}+\frac{z\phi^*}{\xi}\right)v^1\left(-4\mathrm{i}\xi-\frac{2\mathrm{i}\phi\phi^*}{\xi}+\mathrm{i}z\right)v^2. \qquad (6)$$

In matrix form we can set

$$v_\xi=[A_0\xi+A_1+(1/\xi)A_2]v \qquad (7)$$

where

$$A_0=\begin{pmatrix}4\mathrm{i} & 0\\ 0 & -4\mathrm{i}\end{pmatrix}$$
$$A_1=\begin{pmatrix}-\mathrm{i}z & -4\phi\\ -4\phi^* & \mathrm{i}z\end{pmatrix}$$
$$A_2=2\mathrm{i}\begin{pmatrix}\phi\phi^* & -(\phi_z+\frac{1}{2}\mathrm{i}z\phi)\\ (\phi^*_z-\frac{1}{2}\mathrm{i}z\phi^*) & -\phi\phi^*\end{pmatrix}.$$

Equations (5) and (7) will form the basis of the asymptotic expansion that we are going to perform in the next section.

3. Asymptotic expansion

For the construction of the asymptotic expansion, we set, following Wasow [8],

$$v=\exp(a_0\xi^2+a_1\xi)\xi^\mu\sum_k C_k\xi^{-K} \qquad (8)$$

so that

$$v_\xi=[2a_0\xi+a_1+(\mu-k/\xi)]\xi^\mu\exp(a_0\xi^2+a_1\xi)\sum C_k\xi^{-K}.$$

Then from equation (7) we obtain

$$[2a_0\xi+a_1+(\mu-k/\xi)]\exp(a_0\xi^2+a_1\xi)\xi^\mu\sum C_k\xi^{-K}$$
$$=(A_0\xi+A_1+A_2\xi^{-1})\exp(a_0\xi^2+a_1\xi)\xi^\mu\sum C_k\xi^{-K}. \qquad (9)$$

Equating different powers of ξ in (9) we construct equations for C_k, which can be solved to yield the two independent sets of solutions

$$\tilde{v}_\infty(1,z,\xi)\sim\exp(2\mathrm{i}\xi^2-\mathrm{i}z\xi)\left[\begin{pmatrix}1\\0\end{pmatrix}+\xi^{-1}\begin{pmatrix}\phi\phi^*_z-\phi^*\phi_z-\frac{1}{2}\mathrm{i}z\phi\phi^*\\ \frac{1}{2}\mathrm{i}\phi^*\end{pmatrix}+\ldots\right] \qquad (10)$$

$$\tilde{v}_\infty(2,z,\xi)=\exp(-2\mathrm{i}\xi^2+\mathrm{i}z\xi)\left[\begin{pmatrix}0\\1\end{pmatrix}+\xi^{-1}\begin{pmatrix}-\frac{1}{2}\mathrm{i}\phi\\ \phi^*\phi_z-\phi\phi^*_z+\frac{1}{2}\mathrm{i}z\phi\phi^*\end{pmatrix}+\ldots\right]. \qquad (11)$$

At this point it is interesting to note an identity which will be useful later. From equation (4) we note

$$\phi\phi^*_z-\phi^*\phi_z-\tfrac{1}{2}z\phi\phi^*=\tfrac{1}{2}\mathrm{i}\int\phi\phi^*\,\mathrm{d}z+C. \qquad (12)$$

To obtain the asymptotic expansion near $\xi=0$, we put $\xi=1/\eta$ and let $\eta\to\infty$ in equation (7). Then (7) is transformed to

$$V_\eta=-(A_2\eta^{-1}+A_1\eta^{-2}+A_0\eta^{-3})V. \tag{13}$$

We then set

$$V=\eta^\mu\sum C_k\eta^{-K}$$

so that we obtain

$$(\mu-k/\eta)\eta^\mu\sum C_k\eta^{-K}=-(A_2\eta^{-1}+A_1\eta^{-2}+A_0\eta^{-3})\eta^\mu\sum C_k\eta^{-K}.$$

Then the degeneracy condition for C_0 leads to the following equation for μ:

$$\det[A_2+I\mu]=0 \tag{14}$$

from which we can obtain

$$\mu^2=A[(\phi_z+\tfrac{1}{2}\mathrm{i}z\phi)(\phi_z^*-\tfrac{1}{2}\mathrm{i}z\phi^*)-(\phi\phi^*)^2]=2k. \tag{15}$$

However, note that

$$\mathrm{d}\mu^2/\mathrm{d}z=0 \qquad \text{if and only if } \phi\phi^*=\text{constant}. \tag{16a}$$

So the set of solutions near $\xi=0$ is

$$\tilde{v}_0(1,z,\xi)=\mathrm{e}^{u(z)}\xi^{-2k}\left[\begin{pmatrix}(\mathrm{i}\phi_z+\frac{1}{2}\mathrm{i}z\phi)/(\mathrm{i}\phi\phi^*+k)\\ 1\end{pmatrix}+\frac{\xi^{-1}}{1-4k}\begin{pmatrix}x_2\\ y_2\end{pmatrix}+\ldots\right] \tag{16b}$$

where

$$\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\begin{pmatrix}\mathrm{i}(\phi_z+\frac{1}{2}\mathrm{i}z\phi)/(\mathrm{i}\phi\phi^*-k)\\ 1\end{pmatrix}z$$

$$z=-2\mathrm{i}(\phi_z^*-\tfrac{1}{2}\mathrm{i}z\phi^*)\left(4\phi-\frac{z\phi_z+\frac{1}{2}\mathrm{i}z\phi}{\mathrm{i}\phi\phi^*+k}\right)-(2\mathrm{i}\phi\phi^*+2k-1)\left(\mathrm{i}z-4\mathrm{i}\phi^*\frac{\phi_z+\frac{1}{2}\mathrm{i}z\phi}{\mathrm{i}\phi\phi^*+k}\right) \tag{17}$$

and

$$\tilde{v}_0(2,z,\xi)\approx\mathrm{e}^{u^*(z)}\xi^{2k}\left[\begin{pmatrix}1\\ \mathrm{i}(\phi_z+\frac{1}{2}\mathrm{i}z\phi)/(\mathrm{i}\phi\phi^*-k)\end{pmatrix}+\ldots\right] \tag{18}$$

where $u(z)$ is the normalising factor for the solution near $\xi=0$ given as

$$u=\int\phi\,\mathrm{d}z \qquad u^*=\int\phi^*\,\mathrm{d}z.$$

In the above expressions the factors occurring can be simplified if we use equation (15), but we have preferred to keep the general structure. At this point we mention some important features of equations (5) and (6).

(*a*) If $v(1,\xi,z)$ is a solution then $Mv^*(1,\xi^*,z)$ is also a solution.

(*b*) If $M\tilde{v}^*(2,\xi^*z)$ is a solution then $\tilde{v}(2,\xi,z)$ is another solution, where $M=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $v(n,\xi,z)$ denotes the solution vector (v_1,v_2) with $n=1,2$ indicating the first and second type of solution.

4. Regions of growth and decay

The next step in our analysis is the segregation of zones in the complex ξ plane, where the solutions defined in the above section show definite patterns of dominancy or subdominancy. From expressions (10) and (11) we can make some important inferences, shown in table 1.

In figure 1 we have depicted this division of the complex eigenvalue plane into several sectors. Let us recall that the lines in the ξ plane originating from the origin on which the solution is maximally dominant or recessive are called Stokes lines. In our above situation $\arg\xi = \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$ are Stokes lines and the sectors are defined as

$$S_j = \xi, \, || \xi | > \rho \qquad \text{for some } \rho$$

with

$$\tfrac{1}{2}(j-1)\pi \leqslant \arg\xi < \tfrac{1}{2}j\pi \qquad j = 1, 2, \ldots .$$

The anti-Stokes lines are

$$\arg\xi = 0, \tfrac{1}{2}\pi, \pi, \tfrac{3}{2}\pi, 2\pi.$$

Table 1.

$0 \leqslant \arg\xi < \frac{1}{2}\pi$	v^1 large	v^2 small
$\frac{1}{2}\pi \leqslant \arg\xi < \pi$	v^1 small	v^2 large
$\pi \leqslant \arg\xi < \frac{3}{2}\pi$	v^1 large	v^2 small
$\frac{3}{2}\pi \leqslant \arg\xi < 2\pi$	v^1 small	v^2 large
$2\pi \leqslant \arg\xi < 2\pi + \delta$	v^1 large	v^2 small

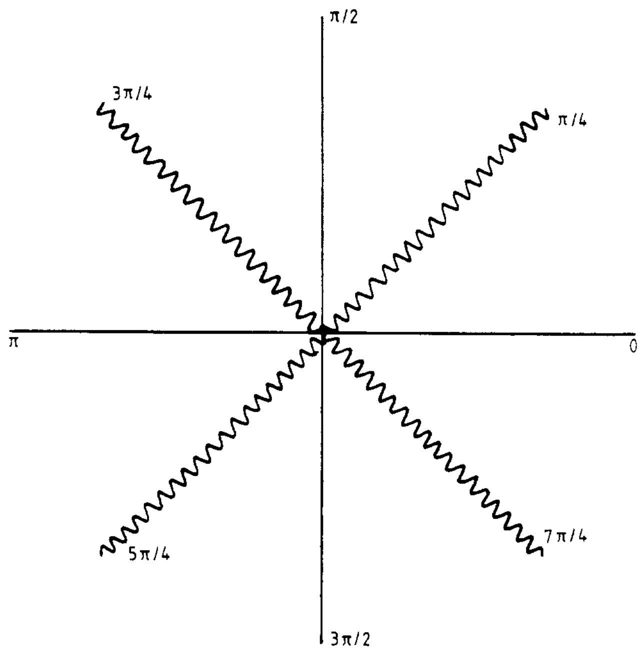

Figure 1. Division of the compex eigenvalue plane into different sectors. Stokes lines (wavy lines) are given by $\arg\xi = \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi$ and $\frac{7}{4}\pi$; anti-Stokes lines (straight lines) are given by $\arg\xi = 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi$.

In the above and also in the whole of the following discussion we have followed the notation detailed below.

$v_j^{(i)}(k, \xi, z)$ denotes a solution of the linear equations (2) and (3), where i denotes the first or second component as above, j denotes the sector and k denotes the type of solution. In general we have two types of solutions for our 2×2 matrix system.

The next important stage is to write down the basic form of the matrix or matrices connecting the solution vectors in several sectors. For this it is important to observe that a solution which was dominating in one sector may become subdominant when its leading terms are cancelled by the contribution from the other component in the other sector. This fact means that the connection matrices are all triangular. Explicitly we have

$$\begin{aligned} v_2 &= v_1\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \\ v_3 &= v_2\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\ v_4 &= v_3\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \\ v_5 &= v_4\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \qquad \text{but } v_5 \equiv v_1. \end{aligned} \tag{19}$$

Now utilising the symmetry properties noted after equation (18) and by recourse to (19) we obtain

$$\begin{aligned} v_2^{(1)} &= v_1^{(1)} + a v_1^{(2)} \\ v_4^{(1)} &= v_3^{(1)} + c v_3^{(2)} \end{aligned} \tag{20}$$

and so

$$c = a \qquad d = b.$$

Also, for $\pi \leqslant \arg \xi < \frac{3}{2}\pi$

$$v_3^{(1)} = b v_1^{(1)} + (1+ab) v_1^{(2)}. \tag{21}$$

For $\frac{3}{2}\pi \leqslant \arg \xi < 2\pi$

$$v_4(\xi, z) = v_3(\xi, z)\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \tag{22}$$

from which it follows that

$$\begin{aligned} v_4^{(1)} &= v_1^{(1)}(1+bc) + (a+c+abc) v_1^{(2)} \\ v_4^{(2)} &= b v_1^{(1)} + (1+ab) v_1^{(2)}. \end{aligned}$$

Now crossing this zone we come back to the first sector again. Hence

$$v_5^{(1)} = v_4^{(1)} = (1+bc) v_1^{(1)} + (a+c+abc) v_1^{(2)}$$

$$\begin{aligned} v_5^{(2)} &= d v_4^{(1)} + v_4^{(2)} \\ &= v_1^{(1)}[d(1+bc)+d] + [1+ab+d(a+c+abc)] v_1^{(2)}. \end{aligned} \tag{23}$$

These relations will be of much use when we connect the solution near the origin to that at infinity.

Now from equations (16b) and (18) we observe that

$$\tilde{v}(1,\xi,z)=v(1,\xi,z)-j\mathrm{i}v(2,\xi,z)\ln\xi \tag{24}$$

where

$$j=\left[2(\phi_z^*-\tfrac{1}{2}\mathrm{i}z\phi^*)\left(4\phi-z\frac{\phi_z^*+\frac{1}{2}\mathrm{i}z\phi}{\mathrm{i}\phi\phi^*+k}\right)-\mathrm{i}(2\mathrm{i}\phi\phi^*+2k-1)\left(\mathrm{i}z-4\mathrm{i}\phi^*\frac{\phi_z+\frac{1}{2}\mathrm{i}z\phi}{\mathrm{i}\phi\phi^*+k}\right)\right]\mathrm{e}^{u-u^*}.$$

The logarithm will disappear if $j=0$, $k=\frac{1}{4}$ with the help of sector relations which we obtain from (24):

$$\begin{aligned}&M\bar{v}(1,\xi\,\mathrm{e}^{-\mathrm{i}\pi},z)=\mathrm{e}^{2\mathrm{i}\pi k};\ \bar{v}(1,\xi,z)-\pi_j\,\mathrm{e}^{-2\pi\mathrm{i}k};\ \bar{v}(2,\xi,z)\\&M\bar{v}(2,\xi,\mathrm{e}^{-\mathrm{i}\pi},z)=\mathrm{e}^{-2\mathrm{i}\pi k};\ \bar{v}(2,\xi,z).\end{aligned} \tag{25}$$

So if in the sector $0\leqslant\arg\xi<2\pi$ the solution is $\bar{v}$ then $\bar{v}(\xi\,\mathrm{e}^{2\pi\mathrm{i}},z)=v(\xi,z)J$ is a fundamental solution in $(2\pi,4\pi)$ where

$$J=\begin{pmatrix}\mathrm{e}^{-4\mathrm{i}\pi k} & 0\\ 2\pi j\,\mathrm{e}^{4\mathrm{i}\pi k} & \mathrm{e}^{4\pi\mathrm{i}k}\end{pmatrix}. \tag{26}$$

It is interesting to note that $\det J=1$ for all K. We now seek the matrix connecting v_0 to v_∞ as

$$v_\infty=v_0A$$

with

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}. \tag{27}$$

Now $\det v_\infty=1$ and

$$\det v_0=\frac{-2\mathrm{i}k(\phi z+\frac{1}{2}\mathrm{i}z\phi)}{(\mathrm{i}\phi\phi^*+k)(\mathrm{i}\phi\phi^*-k)}$$

so that

$$\det A=\frac{(\mathrm{i}\phi\phi^*+k)(\mathrm{i}\phi\phi^*-k)}{-2\mathrm{i}k(\phi_z+\frac{1}{2}\mathrm{i}z\phi)}.$$

Then (a, b, c, d, α, β, γ, δ, $\det v_\infty=1$ and the coefficients of the asymptotic expansions) form the monodromy data for our system.

4.1. Properties of matrix A

Now it follows from equation (27) that

$$\begin{aligned}v_\infty(\xi\,\mathrm{e}^{2\pi\mathrm{i}})&=v_0(\xi\,\mathrm{e}^{2\pi\mathrm{i}})A\\&=v_0(\xi)JA\end{aligned} \tag{28}$$

and in the last sector

$$v_5(\xi\,\mathrm{e}^{2\pi\mathrm{i}})=\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}v_1(\xi\,\mathrm{e}^{2\pi\mathrm{i}})$$

leading to

$$\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}=A^{-1}J^{-1}A. \tag{29}$$

We now set $\xi=\hat{\xi}\,e^{-i\pi}$ in the solution v_∞, apply $M=\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ using relations (20)-(23) and obtain the important and fundamental relations

$$\begin{aligned}Mv^{(1)}&=(\alpha\delta\,e^{2\pi ik}+\alpha\beta\pi j\,e^{-2\pi ik}-\beta\gamma\,e^{-2\pi ik})v^{(1)}\\&\quad+[-2\alpha\gamma\tfrac{1}{2}(e^{2\pi ik}-e^{-2\pi ik})-\alpha^2\pi j\,e^{-2\pi ik}]v^{(2)}\\Mv^{(2)}&=[2\beta\delta\tfrac{1}{2}(e^{2\pi ik}-e^{-2\pi ik}+\beta^2\pi j\,e^{-2\pi ik}]v^{(1)}\\&\quad+(-\beta\gamma\,e^{2\pi ik}-\alpha\beta\pi j\,e^{-2\pi ik}+\delta\alpha\,e^{-2\pi ik})v^{(2)}\end{aligned} \tag{30}$$

from which we deduce

$$\begin{aligned}1&=\alpha\delta\,e^{2\pi ik}+\alpha\beta\pi j\,e^{-2\pi ik}-\beta\gamma\,e^{-2\pi ik}\\a&=-2\alpha\gamma\sin(2\pi k)-\pi j\alpha^2\,e^{-2\pi ik}\\b&=\sin(2\pi k)+\pi j\beta^2\,e^{-2\pi ik}\\1+ab&=-\beta\gamma\,e^{2\pi ik}-\alpha\beta\pi j\,e^{-2\pi ik}+\alpha\delta\,e^{-2\pi ik}.\end{aligned} \tag{31}$$

Until now we have been following the methodology laid down in reference [5]. But, as can be seen clearly, this approach, though it helps us to obtain the properties of the monodromy data, does not allow us to determine the Stokes matrices absolutely. In the next section we show that, by using the method in a paper by Sibuya [6], we can explicitly determine the Stokes matrices which in turn can lead to a complete determination of $(\alpha, \beta, \gamma, \delta)$, the matrix connecting the solutions near zero to that at infinity.

5. Sibuya's approach to Stokes parameters

In reference [6] Sibuya treats the equation

$$\frac{d^2v}{d\xi^2}-(\xi^\mu+a_1\xi^\mu+\ldots+a_{\mu-1}\xi+a_\mu)v=0 \tag{32}$$

under the following assumptions.

(i) The differential equation (32) has a unique solution:

$$v=v_\mu(\xi, a_1, \ldots, a_\mu).$$

(ii) v is an entire function of the parameters $(\xi, a_1, a_2, \ldots, a_\mu)$.

(iii) v admits an asymptotic representation:

$$v\sim\xi^{\gamma\mu}\left(1+\sum_{n=1}^{\infty}B_{\mu,n}\xi^{-n/2}\right)\exp[-iE_\mu(\xi, t)]$$

as ξ tends to infinity in the different sectors, where $E_\mu(\xi, t)$ are represented as

$$E_\mu(\xi, t)\approx\frac{2}{\mu+2}\xi^{(\mu+2)/2}+\sum_{n=1}^{\mu+1}A_{\mu,n}\,\xi^{(\mu+2-n)/2} \tag{33}$$

and r_μ, $A_{\mu,n}$, $B_{\mu,n}$ are polynomials in $(a_1\ldots a_\mu)$.

Now if we put

$$(1 + a_1\xi^{-1} + \ldots + a_\mu\xi^{-\mu})^{1/2} = 1 + \sum_{k=1}^{\infty} b_k\xi^{-k}$$

then the quantities r_μ and $A_{\mu,n}$ are given by

$$r_\mu = \begin{cases} -\mu/4 & \mu \text{ odd} \\ -\mu/4 - b_{\mu/2} + 1 & \mu \text{ even} \end{cases}$$

along with

$$\sum_{n=1}^{\mu+1} A_{\mu,n}\xi^{(\mu+2-n)/2} = \sum_{1\leqslant n\leqslant \mu/2+1} \frac{2}{\mu + 2 - 2n} b_n\xi^{(\mu+2-2n)/2}. \tag{34}$$

(iv) If we choose ϕ such that $\exp[\mathrm{i}(\mu + 2)\phi] = 1$ then the function $v(\hat{\xi}, \mathrm{e}^{\mathrm{i}\phi}a_1 \ldots \mathrm{e}^{\mathrm{i}\phi\mu}a_\mu)$ is also a solution.

With $\theta = \exp[\mathrm{i}(2\pi/\mu + 2)]$ the solution in the jth sector is given as

$$v_j(\xi, t) \sim \theta^{-jr_{\mu-j}}\xi^{r_{\mu_j}}\left(1 + \sum_{n=1}^{\infty} B_{\mu,n,j}\xi^{-n/2}\right) \exp[(-1)^{j+1}\,\mathrm{i}E_\mu(\xi, \alpha)] \tag{35}$$

as $\xi \to \infty$ in the sectors.

(v) The two solutions $v_{\mu_{j+1}}$ and $v_{\mu_{j+2}}$ are linearly independent because $v_{\mu_{j+1}}$ is subdominant in the $(j + 1)$th sector and $v_{\mu_{j+2}}$ is dominant. Therefore v_μ is a linear combination of $v_{j+1}^{(\mu)}$ and $v_{j+2}^{(\mu)}$:

$$v_j(\xi, t) = c_j(t)v_{j+1} + \tilde{c}_j(t)v_{j+2} \tag{36}$$

where c_j, $\tilde{c}_j$ are Stokes multipliers. For $\mu = 2$

$$c_j(a_1, a_2) = \begin{cases} 2^{b_2}\exp\left[\frac{1}{4}a_1^2 - \mathrm{i}\pi\left(\frac{b_2}{2} - \frac{1}{4}\right)\right]\frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2} + b_2)} & j \text{ even} \\ 2^{-b_2}\exp\left[-\frac{1}{4}a_1^2 + \mathrm{i}\pi\left(\frac{b_2}{2} + \frac{1}{4}\right)\right]\frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2} - b_2)} & j \text{ odd} \end{cases}$$

$$\tilde{c}_j = \begin{cases} -\mathrm{i}\exp(-\pi b_2) & j \text{ even} \\ -\mathrm{i}\exp(\pi b_2) & j \text{ odd} \end{cases} \tag{37}$$

$$b_2 = \tfrac{1}{2}a_2 - \tfrac{1}{8}a_1^2.$$

We have quoted the above result to improve the clarity of our case. To apply the above result we first single out the second-order equation by eliminating any one of the components.

By eliminating v_2 we obtain

$$\begin{aligned} v_{1\xi\xi} = \{ &-16\xi^2 + 8z\xi - 4\mathrm{i} - z^2 + \xi^{-1}[8\mathrm{i}(\phi^*\phi_z - \phi\phi_z^*) - 4z\phi\phi^*] \\ &+ \xi^{-2}[-2\mathrm{i}\phi\phi^* - 4\phi^2\phi^{*2} + 4(\phi_z + \tfrac{1}{2}\mathrm{i}z\phi)(\phi_z^* - \tfrac{1}{2}\mathrm{i}z\phi^*)] + \ldots\} \end{aligned} \tag{38}$$

where we have not written down the higher terms as they will not be important for $\xi \to \infty$.

Scaling the variable ξ, v as $2\xi = \xi'$, $4v = v'$ we arrive at

$$v'_{1\xi'\xi'} = [\xi'^2 + z\xi' - (\mathrm{i} - \tfrac{1}{4}z^2) - 4\xi'^{-1}(\phi\phi_z^* - \phi^*\phi_z - \tfrac{1}{2}\mathrm{i}z\phi\phi^*) + \ldots]v'_1. \tag{39}$$

Since this is now a scalar equation we will omit the subscript 1 and call it v. We attach the index 'j' to v as $v_{(j)}$ will denote the solution in the sector identified by the letter j which will be one of those described at the start of § 4. We will now utilise the results of Sibuya for equations of type (39), for which the identification of solutions of (39) with those of (6) are essential. If this is done then equation (36), along with (37), will yield the Stokes parameters. To proceed with the programme outlined above we first switch from vector to matrix notation for the solutions in (10) and (11). That is, the matrix solution is constructed as

$$v^{ij} = \begin{pmatrix} v^{11} & v^{12} \\ v^{21} & v^{22} \end{pmatrix} \tag{40}$$

where

$\begin{pmatrix} v^{11} \\ v^{21} \end{pmatrix}$ is actually $\tilde{v}(1, z, \xi)$ and $\begin{pmatrix} v^{12} \\ v^{22} \end{pmatrix}$ is equivalent to $\tilde{v}(2, z, \xi)$

both in the exact and asymptotic situations. If we affix a subscript j as before then

$$\begin{pmatrix} v^{11}_{(j)} \\ v^{21}_{(j)} \end{pmatrix}$$

will denote the solution in the jth sector. Now in our particular case we have

$$v^{11}_{(1)} = v^{11}_{(0)} + av^{12}_{(0)} \tag{41}$$

and from equation (36)

$$v_{(-1)} = c_{-1}v_{(0)} + \tilde{c}_{-1}v_{(1)} \tag{42}$$

or

$$v_{(1)} = \frac{1}{\tilde{c}_{-1}} v_{(-1)} - \frac{c_{-1}}{\tilde{c}_{-1}} v_{(0)}.$$

But we have the identification $v^{11}_{(0)} = v_{(-1)}$, $v^{11}_{(1)} = v_{(1)}$ and $v^{12}_{(0)} = (-\mathrm{i}\phi)v_{(0)}$. These equations, when coupled with (41) and (42), yield

$$a = \frac{1}{\mathrm{i}\phi} c_{-1} \qquad \tilde{c}_{-1} = 1. \tag{43}$$

Furthermore

$$v^{12}_{(2)} = v^{12}_{(1)} + bv^{11}_{(1)}. \tag{44a}$$

But

$$\left.\begin{aligned} v_{(2)} &= (1/\mathrm{i}\phi)v^{12}_{(1)} \\ v_{(3)} &= v^{11}_{(1)} \end{aligned}\right\} v_{(4)} = v^{12}_{(2)}. \tag{44b}$$

Now

$$v_{(2)} = c_2v_{(3)} - \tilde{c}_2v_{(4)}. \tag{45}$$

Comparing (44a) and (45) with the help of (44b) we obtain

$$b = -\mathrm{i}\phi c_2 \qquad \tilde{c}_2 = -1. \tag{46}$$

Similarly we deduce

$$\left.\begin{aligned} c &= (1/\mathrm{i}\phi)c_{-1} \\ \tilde{c}_{-1} &= 1 \end{aligned}\right\} \qquad \begin{aligned} d &= -\mathrm{i}\phi c_6 \\ \tilde{c}_6 &= -1 \end{aligned} \tag{47}$$

where the c_j are given by equation (37) with

$$b_1 = -\tfrac{1}{2}z \qquad b_2 = \tfrac{1}{2}(\mathrm{i} - \tfrac{1}{2}z^2).$$

It is quite important to observe that our explicit determination of the Stokes parameters respects our earlier derived constraints, $a = c$, $b = d$. Furthermore, if these explicit values are used in (31) then it is, in principle, possible to determine $(\alpha, \beta, \gamma, \delta)$ which was not the case with Flaschka and Newell's method.

6. Properties of the monodromy data

(*a*) The matrix functions v_j are holomorphic in $S_j = \{\xi, |\xi| > 0 \text{ and } \tfrac{1}{2}(j-1)\pi \leqslant \arg \xi < \tfrac{1}{2}\pi j\}$ such that

$$v_j \sim \tilde{v}_j = \left(1 + \frac{c_1}{\xi} + \dots\right)\begin{pmatrix} \mathrm{e}^{\theta} & 0 \\ 0 & \mathrm{e}^{-\theta} \end{pmatrix} \tag{48}$$

$$\theta = 2\mathrm{i}\xi^2 - \mathrm{i}z\xi \qquad \text{as } |\xi| \to \infty \text{ in } S_j$$

and

$$v_{j+1} = v_j A_j.$$

(*b*) A matrix function w of the form

$$w(\xi) = \hat{w}(\xi)\begin{pmatrix} \xi^{-2k} & 0 \\ 0 & \xi^{2k} \end{pmatrix} \tag{49}$$

with $\hat{w}(\xi)$ holomorphic, exists such that for $\xi \in S_1$ $v_1(\xi) = w(\xi)A$ with

$$\det A = \frac{(\mathrm{i}\phi\phi^* + k)(\mathrm{i}\phi\phi^* - k)}{-2\mathrm{i}k(\phi_z + \tfrac{1}{2}\mathrm{i}z\phi)}. \tag{50}$$

(*c*) The solution matrix $v(\xi)$ has the symmetry

$$M\tilde{v}^*(\xi)M = \tilde{v}(\xi) \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{51}$$

(*d*) A_j matrices are independent of z. So differentiating $v_{j+1} = v_j A_j$ with respect to z and multiplying by v_{j+1}^{-1} we obtain

$$\begin{aligned} v_{j+1z}v_{j+1}^{-1} &= v_{jz}A_j v_{j+1}^{-1} \\ &= v_{jz}A_j(v_jA_j)^{-1} = v_{jz}v_j^{-1} \end{aligned} \tag{52}$$

so that $v_{jz}v_j^{-1}$ is well defined in a deleted neighbourhood of $\xi = \infty$ and its asymptotic expansion is that of $\tilde{v}_z\tilde{v}^{-1}$, uniform for $|\xi| > \rho$. Now using (48) and its x derivative we obtain

$$\begin{aligned} \tilde{v}_z v^{-1} &= \mathrm{i}\xi\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{i}\left[c_1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} -\mathrm{i}\xi & \phi \\ \phi^* & \mathrm{i}\xi \end{pmatrix} \end{aligned} \tag{53}$$

which is nothing but the matrix occurring in the L operator pertaining to NLSE.

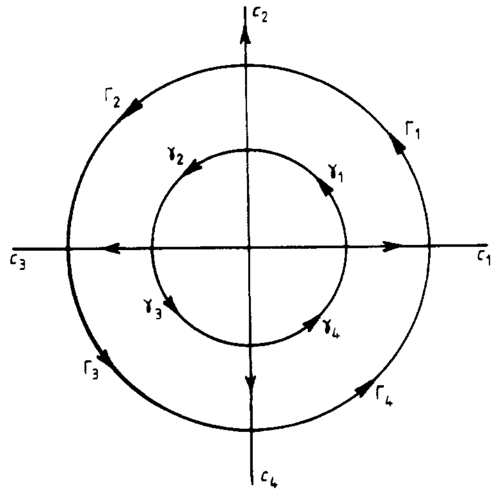

Figure 2. Contours for the integral equations.

Similarly, near $\xi = 0$ we have

$$w_z w^{-1} = \tilde{w}_z \tilde{w}^{-1} = \varphi(\xi)$$

with

$$\tilde{v} = \begin{pmatrix} 1 - \dfrac{iP}{2\xi} & -\dfrac{i\phi}{2\xi} \\ \dfrac{i\phi^*}{2\xi} & 1 + \dfrac{iP}{2\xi} \end{pmatrix} \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{\theta^*} \end{pmatrix}. \tag{54}$$

Evaluating $\tilde{v}_\xi \tilde{v}^{-1}$ we observe that $\tilde{v}_\xi \tilde{v}^{-1}$ is equal to the time part of the Lax operator. Therefore all we need is to demonstrate that the non-linear equation is a result of isomonodromy deformation of the linear problem.

Writing out the contour integrals over the contours shown in figure 2 we can prove that (we do not give the details of the computation, as the method of reference [9] remains almost unaltered)

$$\phi^* = -\lim_{\xi \to \infty} 2i\xi v^1 e^{-\theta} \tag{55}$$

$$\phi = \lim_{\xi \to \infty} 2i\xi v^2 e^{\theta}$$

and finally we obtain

$$\phi = -\frac{1}{\pi} \int_{c_{13}} \exp(4i\xi^2 - 2iz\xi)\, d\xi = \exp(-\tfrac{1}{2}iz^2) \tag{56}$$

which satisfies both of the equations (6) and (7).

7. Conclusions

In this paper we have studied in detail the monodromy problem related to the non-linear Schrödinger equation and Painlevé IV, through similarity variables. Though the general problem of deformation of second- and third-order equations has been studied by the Japanese school, we think that the above analysis helps to clarify any special features that may arise in any particular non-linear problem.

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